Chapter 1

Synchronizability and Connectivity of Discrete Complex Systems

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The synchronization of discrete complex systems is critical in applications such as communication and transportation networks, neuron respiratory systems, and other systems in which either congestion can occur at individual nodes, or system wide synchrony is of importance to proper functionality. The first non-trivial eigenvalue of a network's Laplacian matrix, called the algebraic connectivity, provides a quantifiable measure of synchronizability in a network. We study the general relationship between network topology, clustering coefficient distributions, and synchronizability, as well as the effects of degree preserving rewiring on network synchronizability. In addition, we compare the synchronizability of different network topologies, including Poisson random graphs, geometric networks, preferential attachment networks, and scale-rich networks. We also explore uses of the algebraic connectivity in the design and management of complex networks where synchronization is desired (respiration networks), or detrimental to network performance (router networks).

1.1 Background

In the study of discrete complex systems, it is often important to understand the likelihood of *synchronization* on a given network. For example, synchronization in air transportation networks results in delays and congestion at airports.

Consider the network of nodes (airports, routers, neurons) and edges connecting them (flights, ethernet, axons) that models such a complex system. Denote the state of node i at time t by $x_i(t)$. How do the states of all nodes change over time? Clearly if nodes do not take any note of their neighbors there is no chance for synchronization.

As in [1], we assume that all nodes are identical and conform to the following generic discrete time equation to determine their next state:

$$x_i(t+1) = f(x_i(t)) + \kappa \left[\frac{1}{k_i} \sum_{j|i \leftrightarrow j} f(x_j(t)) - f(x_i(t)) \right] \tag{1.1}$$

where $i \leftrightarrow j$ denotes an edge between i and j [1]. κ is called the *coupling strength* and is a scalar describing the extent to which neighbors effect the state of a node. $f(x)$ is any differentiable function that describes the behavior of a node in the absence of outside influence: notice that if $\kappa = 0$, the equation is simply $x_i(t+1) = f(x_i(t))$.

An initial condition $(x_1(0), \cdots, x_n(0))$ *synchronizes* if for all i, j

$$\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0 \tag{1.2}$$

and we say that the entire network synchronizes if $x_1 = \cdots = x_n$ is an attracting set.

The *Laplacian matrix* of a network is constructed by setting $a_{i,j} = -1$ if $i \leftrightarrow j$ and 0 otherwise for all $i \neq j$, and each $a_{i,i} = deg(i)$. This matrix is positive semi-definite, and all the associated eigenvalue are non-negative real numbers. It has been shown that the second smallest eigenvalue, λ_2, or the *algebraic connectivity* plays a decisive role in determining the synchronizability of a given network in the above sense [1][2][7][8]. A small example is given in figure 1.1.

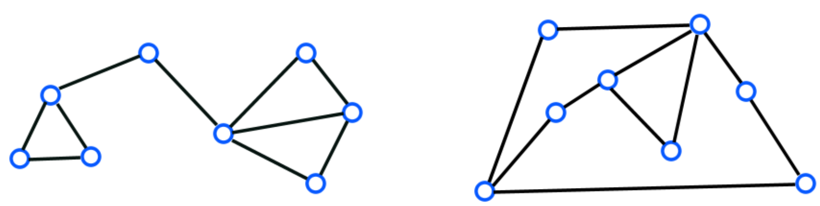

Figure 1.1: These two networks have the same degree sequence, however notice that the network on the left seems weakly connected. Intuitively we expect the network on the right to be more synchronizable. On the left, $\lambda_2 = 0.238$, and on the right $\lambda_2 = 0.925$.

1.2 λ_2 of Erdös-Rényi random graphs

In the Erdös-Rényi, or $G_{n,p}$, construction first defined by Paul Erdös and Alfréd Rényi in [5], we take n nodes and between each pair of nodes create an edge with probability p. This type of network is often referred to as a Poisson random graph, since for large n the degree distribution tends toward a Poisson distribution [4].

Separate constructions were run for $n = \{100, 200, 300\}$ and for $p = \{\frac{2}{n}, \frac{2.5}{n}, \cdots, \frac{5}{n}\}$. The results are shown in figure 1.2 (top).

As one expects, adding more edges to the network (by increasing p) improves synchronizability. As the average degree of the network increases, λ_2 increases exponentially. By reducing the number of tightly coupled clusters in the network and creating a larger loosely connected network, the ability of the network to synchronize is drastically reduced.

1.3 λ_2 of geometric random graphs

Random geometric graphs are created by placing nodes at random coordinated inside a fixed shape, and connecting any nodes within a threshold distance r. Our construction created geometric graphs by randomly assigning (r, θ) polar coordinates to each node inside the unit circle. Separate constructions were run for $n = \{100, 200, 300\}$ and for $r = \{0.25, 0.27, 0.29, 0.31, 0.33\}$. The results are shown in figure 1.2 (center).

Increasing n forces more nodes into the same confined space, so it is not surprising that despite using the same values for r, the networks with 300 nodes have significantly higher average degree than those with fewer nodes. We expect a network of 100 nodes to have the same average degree as a network of 200 nodes when the connection area is double (or equivalently, that the radius is $\sqrt{2}$ longer). As a result, for a given average degree, increasing n forces r to decrease. Since more nodes must be traversed to move across the network, the ability of a network to synchronize is poor. Compared to the Erdös-Rényi graphs, the geometric graph synchronizability is worse for a given average degree. This shows a fundamental difficulty for complex system synchronization where edge construction is based on distance, and demonstrates that a network created by linking closest neighbors will be highly resistant to synchronization.

1.4 λ_2 of preferential attachment graphs

In Barabasi's preferential attachment model, nodes are added one at a time with a fixed number of incoming connections k. Then, for each incoming stub, a random end of an edge is chosen from the existing network, and the node at this end is linked to the new node [3]. In other words, a node with degree 6 is twice as likely to receive one of the new connections as a node of degree 3. We choose to use a slight variation on this basic model, in which the number

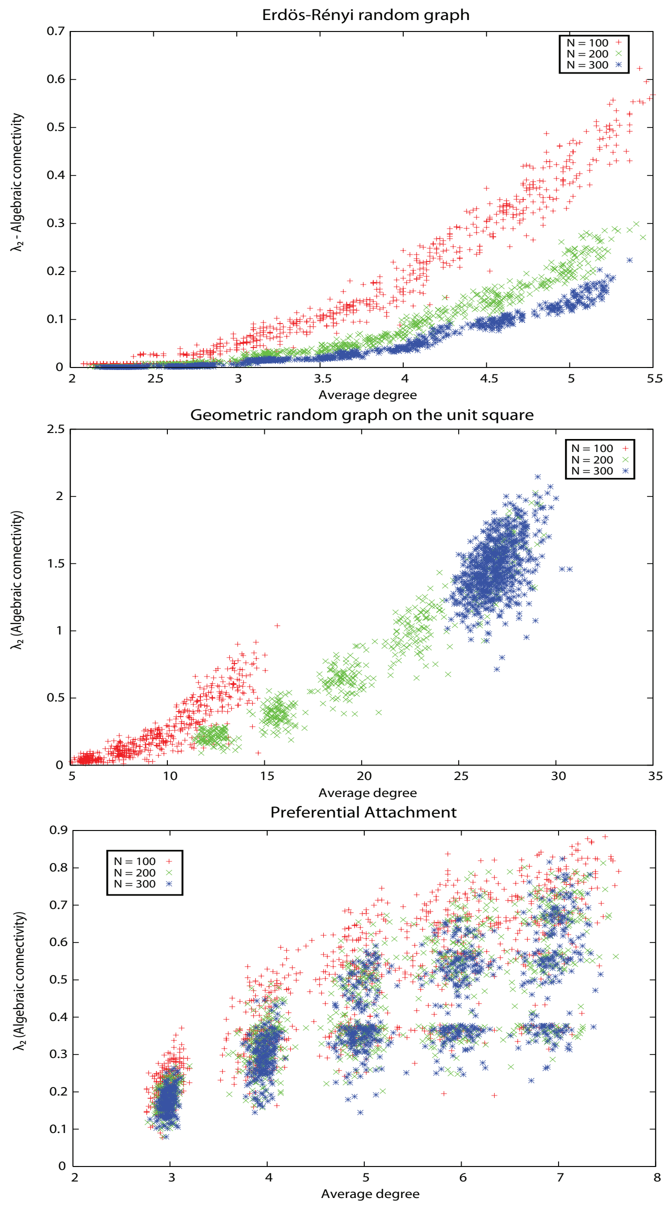

Figure 1.2: Plots depicting the relationship between average degree and λ_2 for randomly generated Erdös-Rényi graphs (top), geometric graphs (center), and preferential attachment graphs (bottom). N is the number of nodes in the network.

of incoming connections for a new node is uniformly random from 1 to k. This results in a wider variety of networks with different average degrees centered around $k+1$. Separate constructions were run for $k = \{2, 3, 4, 5, 6\}$ and for $n = \{100, 200, 300\}$. The results are shown in figure 1.2 (bottom).

We see an unexpected phenomena in the data, resulting in some clustering of datapoints along the y axis, especially just beneath $\lambda_2 = 0.4$. By observing specific networks on opposite sides of this divide, it became clear that this strange behavior was due to the "winner takes all" scenario, in which an early node in the network construction gains an unusually large degree very quickly. New nodes prefer to connect with high degree nodes, so a single node of very high degree causes most new nodes to connect with it. A network which relies on a single highly connected hub is less synchronizable than a network centered around a core of several nodes.

1.5 Neuron rhythmogensis

In mammals, a small group of neurons in the brainstem called the pre-Bötzinger complex is responsible for generating a regular rhythmic output to motor cells that initiate a breath. Disconnected, these neurons are unable to provide enough output to activate the motor neurons, but their interconnected network structure allows them to synchronize without any external influence and produce regular bursts. An example of a typical neuron's output is in figure 1.3.

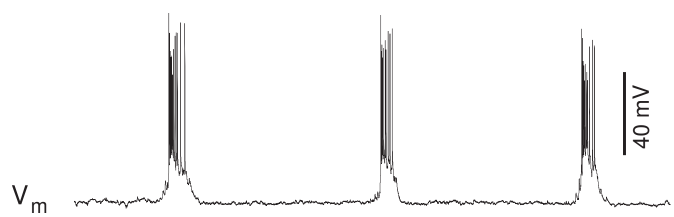

Figure 1.3: Example output from an individual neuron in the PreBötzinger complex. This function corresponds to $f(x)$ in equation 1.1.

Using a detailed simulation written by John Hayes at the College of William and Mary [10], we were able to experiment with how different network topologies control the effectiveness of the pre-Bötzinger complex. Starting from a geometric graph, we used random degree preserving rewiring to sample from all graphs with the same degree distribution, then kept the largest and smallest λ_2 networks found to run the simulation on. The results of the two simulations can be seen in figure 1.4, and provide compelling evidence. We also ran the same experiment starting from a preferential attachment network; however, the results were not as obvious. We used autocorrelation analysis as in [9] to statistically detect better synchronization in the higher λ_2 preferential attachment network. The results are shown in figure 1.5, and confirm that although the difference is undetectable at a glance, the higher λ_2 value statistically shows better synchronization. The difference in autocorrelation is largest during the refractory (non-spiking) period, indicating that the two networks have similar behavior during the spikes, but not between spikes. For more details, see the original paper at [6].

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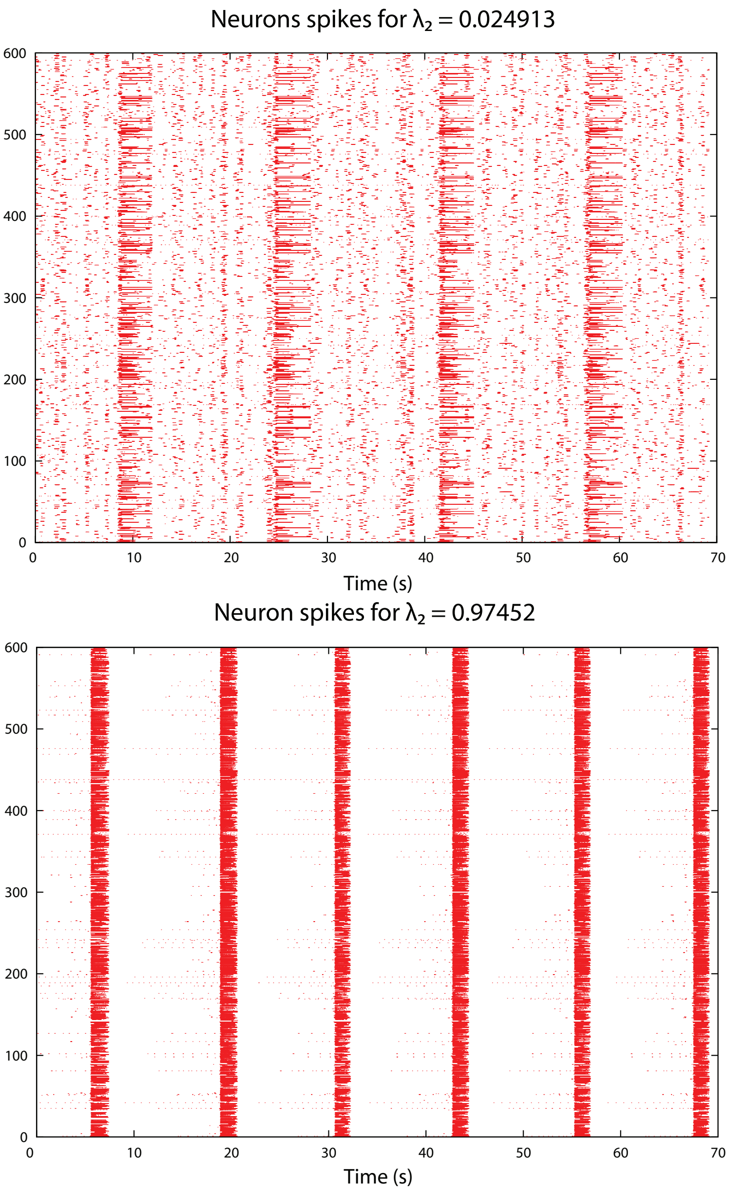

Figure 1.4: Raster plot of neuron output for two sample networks with extreme λ_2 values. A point at (x, y) indicates neuron x is spiking at time y. The higher λ_2 network displays much stronger synchronization amongst all nodes as predicted, as well as a quicker breath frequency.

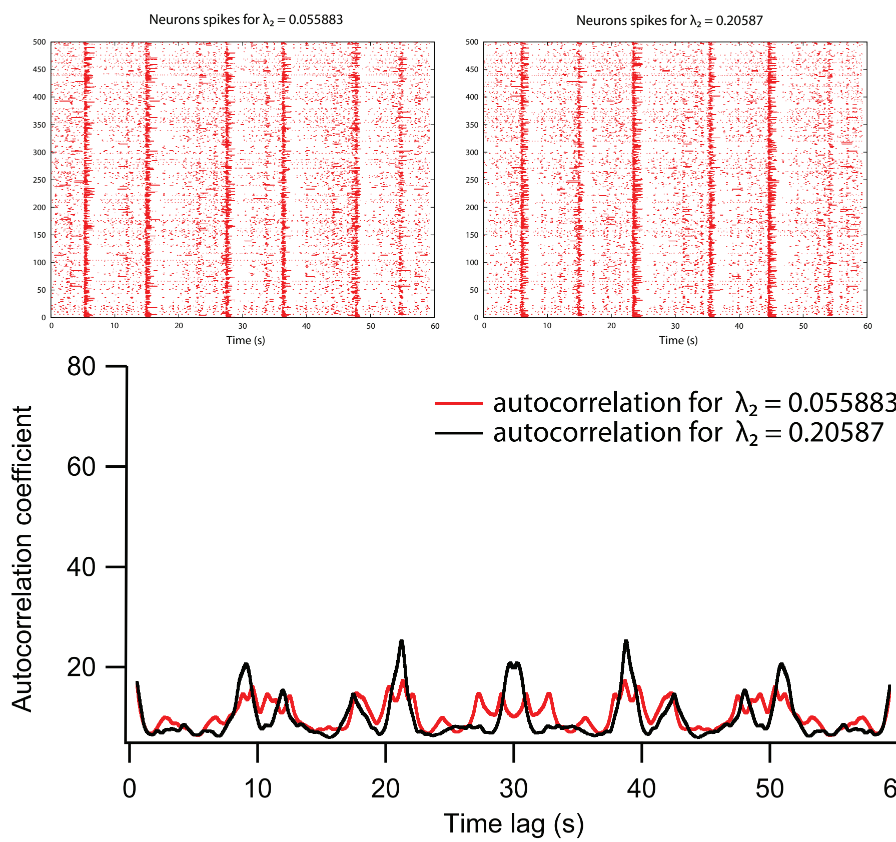

Figure 1.5: An autocorrelation plot of pre-Bötzinger complex synchronization on two networks with the same power-law degree-distribution, but distant λ_2 values. Although the raster plots seem indistinguishable (top two figures), an autocorrelation analysis (bottom) shows that the higher λ_2 network displays statistically better synchronization.